Double integral and reversing the order of integration

Given the function $f(x, y) = x^3 y$ defined on $D = \{(x, y) \in \mathbb{R}^2 \mid 0 < x < 1 \land 0 < y < 1 - x\}$, calculate the value of the integral over the region D. Then, compute the integral again by reversing the order of integration applied.

1 Solution

$$\int_0^1 \int_0^{1-x} x^3 y \, dy \, dx$$

We solve the inner integral:

$$\frac{x^3 y^2}{2}$$

Evaluating at the limits:

$$\frac{(1-x)^2 x^3}{2} - 0 = \frac{(1-2x+x^2)x^3}{2} = \frac{x^3}{2} - x^4 + \frac{x^5}{2}$$

We solve the following integral:

$$\frac{x^4}{8} - \frac{x^5}{5} + \frac{x^6}{12}$$

Evaluating at the limits:

$$\frac{1^4}{8} - \frac{1^5}{5} + \frac{1^6}{12} - 0 = 1/120$$

Now we resolve by reversing the order of integration:

$$\int_0^1 \int_0^{1-y} x^3 y \, dx \, dy$$

We solve the inner integral:

$$\frac{x^4 y}{4}$$

Evaluating at the limits:

$$\frac{(1-y)^4 y}{4} - 0 = \frac{(1-4y+6y^2-4y^3+y^4)y}{4} = \frac{y-4y^2+6y^3-4y^4+y^5}{4}$$

We solve the second integral:

$$\frac{1}{4}\left(\frac{y^2}{2} - \frac{4y^3}{3} + \frac{3y^4}{2} - \frac{4y^5}{5} + \frac{y^6}{6}\right)$$

Evaluating at the limits:

$$\frac{1}{4}\left(\frac{1^2}{2} - \frac{4\cdot 1^3}{3} + \frac{3\cdot 1^4}{2} - \frac{4\cdot 1^5}{5} + \frac{1^6}{6}\right) = 1/120$$